

Entropy, energy and temperature-length inequality for Friedmann universes

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Abstract

In this paper we continue the study of the physical consequences of our modified black hole entropy formula in expanding spacetimes. In particular, we apply the new formula to apparent horizons of Friedmann expanding universes with zero, negative and positive spatial curvature. As a first result, we found that, apart from the static Einstein solution, the only Friedmann spacetimes with constant (zero) internal energy are the ones with zero spatial curvature. This happens because, in the computation of the internal energy U , the contribution due to the non-vanishing Hubble flow must be added to the usual Misner-Sharp energy giving, for zero curvature spacetimes, a zero value for U . This fact does not hold when curvature is present. After analyzing the free energy F , we obtain the correct result that F is stationary only for physical systems in isothermal equilibrium, i.e. a de Sitter expanding universe. This result permits us to trace back a physically reasonable hypothesis concerning the origin of the early and late times de Sitter phase of our universe. Finally, we deduce an interesting temperature-length inequality similar to the time-energy uncertainty of ordinary quantum mechanics but with temperature instead of time coordinate. Remarkably, this relation is independent on the gravitational constant G and can thus be explored also in non gravitational contexts.

Keywords: Bekenstein-Hawking entropy; energy of the universe; apparent horizons; thermodynamics.

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1 Introduction

A great amount of experimental observations during the past decade (see for example [1, 2, 3]) are in agreement with the hypothesis of a present day accelerating universe. The standard Λ CDM cosmological model is described in terms of a spatially flat Friedmann metric endowed with a cosmological constant Λ representing about 70% of the present universe matter-energy content. Despite the enormous success of this model in explaining the cosmological data, some issues have to be further investigated: origin of the dark energy, origin of the inflation, anisotropies of CMBR to cite the most studied ones. Another issue is due to the origin of the entropy of the universe. In the usual view of our universe, the Friedmann equations and the first law of thermodynamics applied to comoving spheres imply an adiabatically expanding universe evolving with constant entropy S . As pointed in [5], our universe does contain a large amount of entropy, and it is unclear how such a very large entropy was created at the big bang or during the primordial inflation. In particular, the actual entropy S_{obs} of the observable universe seems to be of the order $S/k_B \sim 3.1 \times 10^{104}$, where k_B is the Boltzmann constant (see [6] and references therein). The supermassive black holes [6] represent the largest contributor to the entropy of our universe. However, the entropy is a measure of the degrees of freedom of a system and in the calculation of S_{obs} the contribution due to the non static nature of Friedmann universes should be included. Unfortunately, the way to calculate the degrees of freedom of a dynamic universe is not yet available. Irrespective to this estimated value, it seems rather unlikely that such a huge amount of entropy was created soon after the big bang where the proper dimensions of the observable universe was of the order of the Planck length L_P .

A very useful and intriguing way to understand the amount of the entropy of our observable universe is due to the holographic principle (see [7, 8]). According to this principle, the maximum entropy of a given spherical space region is provided by the biggest black hole sitting inside the region. In an expanding dynamical spacetime (see [9] and references therein) the identification of a black hole is not a simple task. In fact, the event horizon becomes a teleological concept non achievable for finite human observers. Fortunately, in references [10, 11] it has been shown that the apparent horizon is a more suitable and manageable tool to identify black holes in an expanding universe. Obviously, the identification between apparent horizons and black holes in a dynamical context is matter of debate and possible drawbacks can emerge (see for example [12, 13, 14] and references therein). First of all, as well known (see for example [12]), apparent horizons

depend on the foliation of the spacetime. However, we are mainly interested to the apparent horizons of Friedmann spacetimes. In this regard, a given comoving observer is equipped with its apparent horizon of a given proper spherical radius L_h , while a different comoving observer sitting in another galaxy at the same cosmic time t will be equipped with a different apparent horizon (a spherical region centered in its galaxy) with proper radius of magnitude L'_h but with $L_h = L'_h$. Hence, any comoving experimenter in the universe agrees with the fact that the universe can be considered as a thermodynamic object at its apparent horizon at the proper areal radius $L_h(t)$. Another possible drawback is due to the fact that dynamical horizons can be also timelike or null (see for example [13]), it substantially depending on the matter equation of state filling the universe. The same happens for apparent horizons in Friedmann spacetimes [4]. Finally (see [14] and references therein), some ambiguity can arise for the definition of the surface gravity and related Hawking temperature for dynamical black holes. However, we stress again that we are interested to the study of the thermodynamics of a Friedmann universe at its apparent horizon, without an embedded black hole.

In this regard, any Friedmann universe with positive energy-density has an apparent horizon. This apparent horizon can be equipped with a well-defined surface gravity expressed in terms of the Kodama vector (see for example [15]). As a consequence of these reasonings, we may apply the holographic principle by supposing that the whole entropy of the universe at the apparent horizon with areal radius L_h is given by the one of a black hole of radius L_h and proper area A_h , i.e the celebrated Bekenstein-Hawking entropy [16, 17] given by $S_{BH} = \frac{k_B A_h}{4L_P^2}$. This formula gives a value for S_{obs} given by $S_{obs}/k_B \simeq 10^{122}$ [7, 8] that is greater than the one estimated in [6] without including the gravitational degrees of freedom due to the expansion of the universe that are not directly detectable from the visible and dark matter content of our universe. In practice, in order to calculate the contribution to S_{obs} given by the dynamical degrees of freedom, a Friedmann universe can be seen 'temporarily' (see [18]) as a black hole at its apparent horizon. More recently (see [19, 15]), the holographic principle has been applied to the apparent horizon in order to 'read' the Friedmann equations as representing exactly the first law of thermodynamics. Moreover, in [20] has been shown that the suitable notion of temperature is the one present in reference [15] and that the apparent horizon is the suitable radius to study the thermodynamics of the universe. It is thus of great importance to study the validity of this identification in an expanding universe and its physical

consequences.

We have recently argued [9, 21] that after using the suitable theorems for the formation of trapped surfaces caused by spherical mass concentration in Friedmann universes (for instance see [22, 23, 24, 25]), the usual formula $S_{BH} = \frac{k_B A}{4L_P^2}$ must be modified. Such a modification includes the Hubble flow contribution to the entropy of the universe. In this paper we continue the investigations present in [9, 21] by analyzing the thermodynamical consequences of our generalized entropy formula. In particular, we explore our formula for the internal energy and the free Helmholtz energy. This allows us to investigate the expansion history of our visible universe in terms of the (on average) temperature of the universe T_u and the holographic temperature T_h at the apparent horizon. This way, we can depict the existence of a pure de Sitter phase of the universe in terms of the difference $T_u - T_h$.

In section 2 we present a derivation of our generalized entropy formula for Friedmann spacetimes. In section 3 we write down the first law of thermodynamics. In section 4 we give the explicit expression of the internal energy U (in terms of a time integration) and the effective pressure acting at the apparent horizon. In section 5 we study the free Helmholtz energy F . In section 6 some reasonings concerning a physically motivated relation between the holographic temperature T_h at the apparent horizon L_h and the 'thermodynamical' temperature of the universe T_u inside L_h are given. Finally, section 7 is devoted to some final remarks and discussions while in the appendix a we analyze our entropy formula for the McVittie solution and in appendix b we study the flat case with a discussion concerning the partition function.

2 Entropy of Friedmann universes

In this paper we adopt the holographic philosophy to calculate the entropy of Friedmann universes. The line element is

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (1)$$

where as usual $k = -1, 0, +1$. In the flat case, we assume $a(t)$ adimensional, while in the other cases $a(t)$ is a length and both k and r are dimensionless. In [21] we have analyzed the flat case and the hyperbolic and closed universes are briefly analyzed in [9]. As it is well known, all Friedmann universes satisfying weak energy condition have an apparent horizon at the proper

areal radius L_h given by

$$L_h = \frac{c}{\sqrt{H^2 + \frac{kc^2}{a(t)^2}}}. \quad (2)$$

Only for the Milne solution without any matter content and $k = -1$ we have $L_h = \infty$.

The core of the holographic principle [7, 8] is that the maximum entropy S available in a region of size R and area A is the one of a black hole of the same dimensions, i.e. the largest black hole fitting inside. Since the celebrated Bekenstein-Hawking (BH) formula [16, 17], it is customary to pose $S \leq S_{max} = S_{BH} = \frac{k_B A}{4L_P^2}$. This expression can be derived thanks to the Bekenstein bound or the entropy bound (see [9] and references therein): there exists an universal bound for the entropy S of a spherical object of radius R and mass-energy E given by $S \leq S_{max} = \frac{2\pi k_B R E}{hc}$. By setting $E = c^4 R / (2G)$, the BH formula arises. Hence, in the static asymptotically flat case the bound is saturated with $E = M c^2$, where M is the ADM (total) mass of the black hole. In the literature (see for example [5, 19, 15]) the usual expression $S \sim A/4$ is also used in a cosmological non-static context. This line of research is motivated by the request to recast the Friedmann equations as the first law of thermodynamics. However, it should be stressed that the expression $E = c^4 R / (2G)$ is certainly suitable for static black holes in asymptotically flat spacetimes, but it is rather questionable in a cosmological context. Physical intuition suggests that in an expanding universe as the one we live, it is more difficult to build a black hole since the expansion is in opposition to the mass-concentration. In fact, theorems quoted in [22, 23, 24, 25] confirm the physical intuition. These theorems are the starting point of our study. Thanks to the holographic principle, it is natural to suppose (see also [9]) that the entropy bound can be exactly saturated at the apparent horizon (see also [19, 15]) given by (2). In practice, a Friedmann universe is temporarily a black hole [18] at its apparent horizon. We must take a Friedmann solution and wonder what is the mass concentration δM that leads to the formation of a trapped surface. The theorems present in [22, 23, 24, 25] are exactly what we need.

We denote with Σ the spherical three-dimensional hypersurface obtained at $t = \text{const}$. The initial data set g_{ab}, K_{ab}, ρ are given on Σ , g_{ab} is the metric, K_{ab} the extrinsic curvature and ρ the energy-density of the spherically perturbed Friedmann flat spacetime on the three-dimensional space Σ by means of the mass excess δM . We also suppose that the matter current J_a flowing through Σ is vanishing. Suppose that $K_a^a = \text{const}$, i.e. the rate of

change of the volume of Σ is not perturbed: if for a sphere S with proper mass excess δM we have

$$\frac{G}{c^2}\delta M < \frac{L}{2} + \frac{AH}{4\pi c} - k\frac{3V}{8\pi a(t)^2}, \quad (3)$$

then S is not trapped. After applying the entropy bound but with $E = \delta M c^2$ and δM given by the second member of (3) we obtain for the entropy S_h at L_h (see [9])

$$S_h = \frac{k_B A_h}{4L_P^2} + \frac{3k_B}{2cL_P^2} V_h H - \frac{3kk_B}{4L_P^2} \frac{L_h V_h}{a(t)^2}, \quad (4)$$

where as usual (see [26, 27]) we take, also for $k = -1, +1$, $V_h = 4\pi L_h^3/3$. The application of (4) to flat and hyperbolic expanding spacetimes gives no problem (see also appendix a for an application to the McVittie solution).

Some issue can arise when one applies the formula (4) for closed universes with $k = +1$ that are contracting $H < 0$. In particular, when $L_h = a(t)$ we have $S_{BH} = 0$ and the entropy becomes negative when $H < 0$. The same phenomenon appears in [28], in the context of the Cardy-Verlinde formula. This fact, as pointed in [28], could indicate a problem for the entropy bound for closed universes near the turning point where $H_{min} = 0$ and for $H < 0$. Note also that for $k = +1$ and near the turning point at $L_h = a(t)$, the leading term of (4) is exactly the one given by the Cardy-Verlinde formula, provided that the bound (see [28]) $S \leq S_{max} = \frac{2\pi k_B R E}{3hc}$ (in the ordinary spacetime with dimensions $D = 3 + 1$) is used. In our context, as noticed in [25], trapped surfaces naturally arise in a closed universe at the so called south pole and everywhere in the contracting phase, independently on the mass excess δM . It is therefore evident that in these cases the inequality (3) can be satisfied also for $\delta M < 0$ and this makes our procedure to obtain the expression (4) questionable. To avoid these shortcomings, we apply our new formula (4) to expanding universes with $k = -1, 0$ (independently on their matter content) and to closed monotonic ones, for example equipped with a positive cosmological constant, or to a cyclic universe in the expanding phase but with $L_h \ll a(t)$.

Note that for the derivation of (4) in expanding universes it is assumed $a_{,t}/a(t) \neq 0$, i.e. it applies to dynamical universes. Hence, to obtain the static limit we must impose from the onset $H = 0$ and as a consequence the entropy of a static de Sitter universe looks like the usual expression $S \sim A/4$.

We must specify the expression of the temperature T_{D_h} at the apparent horizon. The usual expression is the one proportional to the surface gravity

parameter of a Kodama vector field (see [9]):

$$T_{D_h} = \frac{\hbar c}{2\pi k_B L_h} \left| 1 - \frac{L_{h,t}}{2H L_h} \right|. \quad (5)$$

However, according to the usual thermodynamic, we have $(\frac{\partial S}{\partial U})_{V_h} = 1/T$ and thus the 'thermodynamic' version of (5) must be formally calculated at fixed L_h . As a consequence, the temperature appropriate in our context is

$$T_h = \frac{\hbar c}{2\pi k_B L_h}. \quad (6)$$

We denote the temperature T_h as the holographic temperature at L_h .

3 First law of thermodynamics

We are in the position to derive the first law of thermodynamics. As customary, we can obtain the first law from the finite expression (4) by calculating the expression $T_h dS_h$ where (see [9])

$$T_h dS_h = dU_h + P_h dV_h \quad (7)$$

where T_h is the temperature of the apparent horizon, U_h denotes the internal energy and P_h the effective holographic pressure acting on L_h . Since [19, 15, 26], the apparent horizon is dynamical, the expression for P_h is not generally expected to be the one p effectively present in the energy momentum tensor, i.e. $|P_h| \neq |p|$. In the usual picture [15], the Friedmann equations are equivalent to (7) by setting $S_h \sim A_h/4$, $U_h = E_{ms}$ where E_{ms} is the Misner-Sharp energy and $P_h = (p - \rho c^2)/2$. Note that in that case $P_h = 0$ for stiff matter (although the apparent horizon is dynamical also in this case) and the work term of the visible universe is negative for expanding universes filled with ordinary matter satisfying the dominant energy condition. In the following we show that this picture is modified thanks to the new physically motivated formula (4).

To start with, note that irrespective of the fact that in the Friedmann case all the above quantities depend only on the cosmic comoving time t , we use the practical rule to identify the terms proportional to dV_h as P_h and the remaining terms to dU_h . It should also be noticed that after the transformation $T_h \rightarrow \beta T_h$, $U_h \rightarrow \beta U_h$ and $P_h \rightarrow \beta P_h$, where β is a general constant, the equation (7) is left unchanged. We can advantage of this fact

to choose the suitable normalization factor. For an expanding non-static universe we fix $\beta = 1/2$. We obtain

$$dU_h = \frac{c^4}{2G} dL_h + \frac{c^3}{2G} L_h^2 dH + \frac{kc^4}{2G} \frac{L_h^3}{a^3} da, \quad (8)$$

$$P_h = -\frac{kc^4}{4\pi Ga^2} + \frac{3c^3 H}{8\pi GL_h}, \quad T_h = \frac{\hbar c}{4\pi k_B L_h}. \quad (9)$$

Thanks to this choice, the first term in (8) is nothing else but the Misner-Sharp energy $E_{ms} = c^2 M_{ms}$, where M_{ms} is the mass content within the apparent horizon. Moreover, we expect that in the de Sitter case with $k = 0$, since $L_h = \text{const} = c/H$, the effective pressure P_h must be, apart from the sign, of the same value of the one present in the energy momentum tensor $T_{\mu\nu}$, i.e. $P_h = |-\Lambda c^4/(8\pi G)|$ ¹.

These facts justify this choice. It is also interesting to analyze the static limit for $H = 0$ in (9): this represents the static Einstein universe. In this case we expect P_h to be the effective pressure of this model i.e. $P_h = P_{eff} = p - \frac{\Lambda c^4}{8\pi G}$ where p is the hydrostatic pressure present in $T_{\mu\nu}$ and P_{eff} the effective pressure acting in presence of a cosmological constant in the static universe. In this case P_h is negative for $k = +1$ (the pressure must be negative to contrast the positive curvature) and obviously positive in the hyperbolic case. It is easy to see that by choosing the normalization factor $\beta = 1/4$ we have $P_h = \frac{-kc^4}{8\pi Ga^2}$, $a = \text{const}$ and as a consequence by using $3k/(8\pi Ga^2) = \rho_{eff} = \rho + \Lambda c^2/(8\pi G)$, the other relevant Friedmann equation $4\pi G/3(3P_{eff} + \rho_{eff}c^2) = 0$ is satisfied.

4 The internal energy at the apparent horizon

Once the quantities P_h, U_h, E_{ms}, T_h have been identified, we can write the entropy formula (4) in terms of these quantities. In fact, after calculating $T_h S_h$ (with $\beta = 1/2$), we have:

$$T_h S_h = \frac{E_{ms}}{2} + P_h V_h + \frac{kc^4 L_h^3}{16Ga^2}. \quad (10)$$

Note that the expression (10) generally does not assume the usual form $TS = U + PV$.

¹Remember that the work term pdV in the first law satisfies the usual rule: it is positive if the work is done by the system, while is negative if the work is 'suffered' from the system.

4.1 The flat case

First of all, we analyze in more detail the flat case $k = 0$ obtaining $T_h S_h = E_{ms}/2 + P_h V_h$, that in the static case $H = 0$ reduces to the usual black hole relation $T_h S_h = E_{ms}/2$. With $\beta = 1$ we obtain $T_h S_h = E_{ms} + P_h V_h$ that is identical to the Euler's relation provided that $E_{ms} = U_h$. In the usual approach this is the key assumption, but in our new approach $E_{ms} \neq U_h$. This happens because the gravitational contribution due to the expansion of the universe must be included in U_h . As a consequence of our assumptions, by putting $k = 0$ in (8) we obtain, as calculated in [9], $dU_h = 0$, i.e. $U_h = \text{const.}$ As noticed in [9], from dimensional arguments we have $U_h = 0$. This result is in agreement with an old conjecture [29] concerning the origin of the universe. In the flat case, the added term in (4) proportional to HV takes into account the gravitational degrees of freedom due to the expansion of the universe. The term $\sim dH$ in (8) is the contribution to the internal energy due to the non-static nature of our universe that in the flat case conspires with the energy of the matter to obtain a zero energy for the universe. The internal energy U_h is the energy necessary to create the system and thus it is not a surprise that in a spatially flat system this energy is zero, since our universe can be emerged from a minkowski spacetime.

Concerning the enthalpy \tilde{H} we have, by definition, $\tilde{H}_h = U_h + P_h V_h$. Since $U_h = 0$, from (9) we obtain $\tilde{H}_h = P_h V_h = \frac{c^4}{2G} L_h = E_{ms}$. As a consequence, in the flat case, the enthalpy is nothing else but the Misner-Sharp energy (rather than the internal energy U_h). Furthermore, since $U_h = E_{ms} + E_{grav} = 0$, we can compute the gravitational expansion energy E_{grav} as $E_{grav} = -P_h V_h$. In the de Sitter case, thanks to the reasonings at the end of the section 3, we have $P_h = -p = \Lambda c^4/(8\pi G)$ and $pV_h = -E_{ms}$, i.e. $p = -\rho c^2$, the correct result.

4.2 General case

In this subsection we analyze the most general case with $k \neq 0$, where dU_h is not more identically zero. After writing the (8) as a time derivative we have:

$$U_{h,t} = \frac{c^4}{2G} L_{h,t} + \frac{c^3}{2G} L_h^2 H_{,t} + \frac{kc^4}{2G} \frac{L_h^3}{a^3} a_{,t}. \quad (11)$$

The internal energy (11) is composed of three parts: the first is the usual Misner-Sharp contribution, the second can be associated to the expansion energy of the universe while the third term is the contribution due to the non-vanishing spatial curvature. The curvature k enters in all terms and

conspires to give a non vanishing value for U_h . As pointed in [9], this is a reasonable result since for an expanding universe with $k = -1$ we have the 'creation' of new negatively curved space, i.e. the creation of positive energy. Conversely, to an expanding universe with $k = +1$ it is associated the creation of positively curved space, i.e. the creation of negative energy (strong attractive regime). This interpretation is enforced by the fact that the Friedmann equations can be written in terms of the density parameters $\Omega_m = \frac{8\pi G\rho}{3H^2}$, $\Omega_\Lambda = \frac{\Lambda}{3H^2}$, $\Omega_k = -\frac{kc^2}{a^2H^2}$:

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1. \quad (12)$$

Friedmann equations permit us to associate to the curvature term $-kc^2/a^2$ a binding energy described by the third term $\sim k$ in (11).

As stated above, in the flat case $U_h = 0$. It is interesting to explore the existence of other spacetimes where the internal energy is conserved. From (11) we have

$$U_{h,t} = \frac{c^5 \left[H_{,t} \left(\sqrt{H^2 + \frac{kc^2}{a^2}} - H \right) + 2c^2 k a_{,t} \right]}{2Ga^3 \left(H^2 + \frac{kc^2}{a^2} \right)^{\frac{3}{2}}}. \quad (13)$$

Note that the expression (13) is not defined in the Milne case where the denominator is vanishing, implying $L_h = \infty$. Apart from this case, we impose the equation $U_{h,t} = 0$. From (13) we deduce

$$H_{,t} \left(\sqrt{H^2 + \frac{kc^2}{a^2}} - H \right) + 2c^2 k a_{,t} = 0. \quad (14)$$

To solve equation (14) we use a simple argument. In a Friedmann context both H and a are time functions only. Hence in a monotonic (also decreasing with $H < 0$) phase we have $H = H(t(a))$. Equation (14) becomes, with the physical requirement $a > 0$,

$$H_{,a} = -\frac{2c^2 k}{a^2 \left[\sqrt{a^2 H^2 + kc^2} - aH \right]}. \quad (15)$$

By taking a general expression for $H(a) = G(a)$, both sides of (15) are functions of a only and the differential equation becomes an algebraic constraint for a , i.e. $a = \text{const}$, but necessarily $H = 0$. We thus have the static Einstein universe with $H = 0, a = \text{const}$ with matter content such that

$4\pi G(c^2\rho + 3p) - \Lambda c^4 = 0$. Thanks to these calculations, the total internal energy of this spacetime is zero.

By inspection of (15), the only time dependent solution is $H = Q/a, Q = \text{const.}$ After putting this solution in (15), it is easy to see that no solution arises with $a > 0$. Hence, apart from the static Einstein solution, we found that the spatially flat solutions are the only Friedmann solutions with vanishing internal energy. In particular, the universe perhaps we live on average has zero total energy. When a non-vanishing curvature is present, this curvature gives a contribution to the energy that is negative for $k > 0$ (strong attractive regime, negative binding energy) and positive for $k < 0$ (positive binding energy, repulsive regime).

Consider now Friedmann universes filled with an ordinary perfect fluid together with a positive cosmological constant Λ . As well known, these solutions admit an asymptotic behaviour given by the de Sitter metric. From (13) we can calculate the leading term at late times for $t \rightarrow \infty$:

$$U_h = -\frac{kc^7}{2Ga^2H_\Lambda^3} + o(1/a^2), \quad H_\Lambda = c\sqrt{\frac{\Lambda}{3}}, \quad (16)$$

where the integration constant has been set to zero since the energy of the de Sitter universe is zero. Note that by defining the density ρ_k by $c^2\rho_k V_h = U_k$, from (16) we have for $t \rightarrow \infty$

$$\rho_k \rightarrow -\frac{3kc^4}{8\pi Ga^2} \quad (17)$$

that is exactly the density related to Ω_k in (12).

In practice, from (16) we see that this asymptotic behaviour looks like the binding term $-kc^2/a^2$ present in the Friedmann equation (12). We thus obtain that Friedmann universes equipped with a positive cosmological constant all have an internal energy asymptotically (from the below for $k = -1$ and from the above for $k = +1$) approaching a zero value.

5 The free Helmholtz energy

The Helmholtz energy $F = U - TS$ is a thermodynamic function that, in ordinary systems, is stationary for all systems in thermal equilibrium with a given 'reservoir'. The stationary points of F (at constant volume) represent isothermal thermodynamics equilibrium states. As well known [30, 31], the Helmholtz energy can be defined also in nonequilibrium states. In the usual picture where $U = E_{ms}$, $S \sim A/4$ and T is given by (6), we found $F = 0$. If

this were the case, any Friedmann universe should be in thermal equilibrium at the apparent horizon, but we expect that $F_{,t} = 0$ for $T = \text{const}$ at $V = \text{const}$: in the cosmological context this implies $L_h = \text{const}$, i.e. a Sitter universe. In the following we investigate the expression for F_h arising from our formulas. We obtain:

$$F_h = U_h - \frac{c^4 L_h}{4G} - \frac{c^3 L_h^2 H}{2G} + \frac{kc^4 L_h^3}{4Ga^2}, \quad (18)$$

with U_h given by (11) after a time integration. We study the monotony of F_h . For $F_{h,t}$ we obtain:

$$F_{h,t} = \frac{c^3}{G} L_{h,t} \left[\frac{c}{4} - H L_h + \frac{3ck L_h^2}{4a^2} \right]. \quad (19)$$

From (19) it follows that $F_{h,t} = 0$ for $L_h = \text{constant}$, i.e. for a de Sitter spacetime or for spacetimes equipped with a positive cosmological constant asymptotically approaching a de Sitter phase. First of all, for an ordinary expanding spacetime $L_{h,t} > 0$, while for a de Sitter spacetime $L_{h,t} = 0$. Only spacetimes equipped with a 'phantom' equation of state ($p = wc^2\rho$, $w < -1$) have $L_{h,t} < 0$ ($L_{h,t} \sim (1+w)/\sqrt{\rho}$). In the following, we assume universes filled with a non-phantom fluid. Moreover, note that both for $k = 0$ and $k = -1$ certainly the expression between square brackets is always strictly negative. We thus obtain that for $k = 0, -1$ F_h is always a non-increasing time function and is stationary only when a de Sitter phase is reached.

Concerning the case with $k = +1$, we note that the expression in square brackets is vanishing only for $Ha = 2c/\sqrt{5}$. Moreover, F_h is decreasing for $Ha > 2c/\sqrt{5}$ and increasing for $Ha < 2c/\sqrt{5}$. However, we expect that our calculations for $k = +1$ (and also the entropy bound, see [28]) to obtain S_h are certainly valid for $L_h \ll a$ and in this range certainly we have $F_{h,t} < 0$. Also for $k = +1$ but with $\Lambda > 0$ the term in brackets is expected to be negative. As a result, also for the closed case as far as our assumptions are certainly valid, the free energy is monotonically decreasing and stationary only when $L_h = \text{const}$. These results are in agreement with the physical meaning of the free energy.

6 Holographic and thermodynamic temperature of the universe

In the sections above we have studied the entropy, the internal energy and the free energy of the universe by invoking the holographic principle. In this

context, the 'holographic' temperature T_h at the apparent horizon arises. However, this holographic temperature is not the one T_u of the universe present inside the apparent horizon. It is clearly not so obvious to define an average temperature T_u for the whole universe. Since the highly homogeneous and isotropic matter-energy content of the universe is provided by the CMBR, we may identify with good approximation T_u with the temperature T of the CMBR. At the present epoch $T_0 \simeq 2.72 \text{ K}$ and it scales in the following way: $T_u(t) = T_0/a(t)$. In the following, we investigate the physical relation between T_h and T_u . To this purpose, the fundamental result is the one obtained in section 5, equation (18): the free Helmholtz energy is stationary only when a de Sitter phase is reached, i.e. $L_h = \text{const.}$ Since the free energy is stationary only for systems in thermal equilibrium, during a de Sitter phase for the holographic T_h and thermodynamic temperature T_u we must have

$$T_h = \frac{\hbar c}{4\pi k_B L_\Lambda} = T_u, \quad L_\Lambda = \sqrt{\frac{3}{\Lambda}}, \quad (20)$$

for any positive cosmological constant Λ . This means that to a cosmological constant must be associated a non-vanishing temperature. This temperature is vanishing in the limit $L_h \rightarrow \infty$ and its non zero value is a finite size effect induced by the finite thermodynamic radius L_h of our universe. By continuing on this line of reasoning, according to physical intuition and to the physics of an ordinary gas, it is natural to suppose that when $T_u > T_h$ the apparent horizon of our universe is expanding (i.e. $p = wc^2\rho, w > -1$), while when $T_u < T_h$ we have a phantom (cold) universe ($w < -1$) with a decreasing apparent horizon, i.e. when $T_u > T_h$ we expect an expanding apparent horizon with $L_{h,t} > 0$. Conversely, when $T_u < T_h$ we expect a contracting apparent horizon with $L_{h,t} < 0$. Soon after the big bang near the Planck era we expect $T_u - T_h > 0$, i.e an expanding universe with deceleration parameter $q > -1$. After the Planck era, the effective temperature of the universe T_u quickly decreases until $T_u - T_h \simeq 0$ and a pure de Sitter inflationary era begins. At late times, an asymptotically pure de Sitter phase arises for $T_u \rightarrow T_h$. This picture can be verified by numerical estimations.

At the present epoch, we have $T_{0u} \simeq T_{0CMBR} \simeq 2.72 \text{ K}$, while $T_{0h} \simeq 1.76 \times 10^{-29} \text{ K}$ and thus $T_u > T_h$. Asymptotically we have $T_h = T_u \simeq 1.2 \times 10^{-29} \text{ K} = T_\Lambda$ and the so called thermal death of the universe does not arise in our view.

At the Planck era ($t \simeq t_P$) it is natural to suppose $T_u = T_P = \sqrt{\hbar c^5/(Gk_B^2)}$. Below the Planck time the notion of classical manifold breaks down. We may

for example suppose a non-commutative structure of the spacetime (quantum spacetime [32]). In this context, it is expected (see [33, 34]) that a maximum finite value for H is attained of the order of $H_{max} \sim 1/(\sqrt{3}t_P)$ and we expect that the inequality $T_u > T_h$ still holds. At the Planck era and before inflation the matter content of the universe can be represented by a radiation field at the temperature T_u scaling as $T_u = T_i/a(t)$ in a spatially flat Friedmann universe with $a(t) = \sqrt{\frac{t}{t_i}}$. By taking for a practical estimation for t_i and T_i the actual time and the actual temperature of the CMBR (T_i and t_i must be calculated at the same cosmic time) with $H = 1/(2t)$ we have that the inequality

$$\frac{\hbar H}{4\pi k_B} \leq \frac{T_0}{a(t)} \quad (21)$$

is satisfied for $t \geq t^* \simeq 0.2t_P$. Thanks to the non commutative effects below the Planck time, we have that the maximum value for H_{max} is reached for $t \simeq 0.86t_P$, i.e. we can reasonable assume the validity of the assumption $T_u - T_h \geq 0 \forall t > 0$.

At the beginning of the primordial inflation with $H = H_I$, we must have $T_u = T_{hI} = \frac{\hbar H_I}{4\pi k_B}$. After an unknown reheating mechanism the inflation ends and $T_u > T_h$ up to late times where $T_u \rightarrow T_h$.

Concerning T_u , one may think that the main contribution to T_u is due to the CMBR radiation at the temperature $T = T_0/a(t)$ with density ρ_r , with a correction given by the temperature T_Λ of the cosmological constant Λ . Moreover, we label with T_i all the remaining contributions to T_u with density ρ_i (for example the contributions from magnetic fields, neutrinos, axions...). At first look and for practical purposes, we can consider the universe as a non decoupled mixture between the radiation ρ_r and the dark components of the universe $\rho_\Lambda = \Lambda c^2/(8\pi G), \rho_i$. We thus have within the apparent radius L_h

$$T_u = \frac{\rho_\Lambda T_\Lambda + \rho_r T_r + \rho_i T_i}{\rho_\Lambda + \rho_r + \rho_i}. \quad (22)$$

For universes with a positive cosmological constant, we expect that asymptotically $\{\rho_r, \rho_i\} \rightarrow 0$ and as a result $T_u \rightarrow T_h \rightarrow T_\Lambda = \hbar\sqrt{\Lambda}/(4\pi k_B\sqrt{3})$.

An interesting consequence of our formulas is that to a cosmological constant in an expanding universe can be attributed a non-vanishing temperature T_Λ . This happens because for an expanding universe filled with non-phantom matter-energy, the free energy is not increasing and is stationary only when a de Sitter phase is reached. Since the free energy is stationary only for systems in thermal equilibrium, a de Sitter non-static universe can be depicted as a thermodynamical system such that the universe at L_h is

in thermal equilibrium at the temperature T_u with the 'reservoir' at the holographic temperature T_h with $T_u = T_h$.

Summarizing, suppose that we have an expanding Friedmann universe filled with a certain set $\{s\}$ of matter-energy fluids with equation of state $p_s = w_s \rho_s c^2$, $w_s \geq -1$ and with temperature T_s . Also suppose that the different kinds of matter present in the universe are not decoupled and as a consequence a mean temperature T_u for the universe can be defined. Hence, for the whole history of such a universe we have $T_u \geq T_h$, where

$$T_u = \frac{\sum_s \rho_s T_s}{\sum_s \rho_s} \geq \frac{\hbar c}{4\pi k_B L_h}, \quad (23)$$

We can manipulate the formula (23) in the following way. For the homogeneity and isotropy of Friedmann universes, all the matter-energy density components of the universes are time functions only. To this purpose, we denote with E_{ms}^s the general Misner-Sharp mass of any given component of the universe at L_h .

After using the Friedmann equation $H^2 + kc^2/a^2 = 8\pi G/3 \sum_s \rho_s$, the (23) becomes

$$\sum_s T_s E_{ms}^s \geq \frac{\hbar c^5}{8\pi k_B G}. \quad (24)$$

Equation (24) looks like the usual time-energy inequality of ordinary quantum mechanics but with the temperatures instead of the time, while inequality (23) as a kind of temperature-length uncertainty relation. The physical meaning of (24) (an also of (23)) is very simple. Suppose for simplicity a universe dominated by a single matter energy component 'k'. If the Misner-Sharp mass of this component were very low, its temperature should be huge. In effect, near the big bang we expect $E_{ms}^k \rightarrow 0$ and $T_k \rightarrow \infty$. It should be noticed that at the big bang $S_u = 0$ and $S_{u,t} \geq 0$ for non-phantom universes and as a consequence the second law of thermodynamic is satisfied for our universe at the apparent horizon, as well known, only for non-phantom universes. In this regard, the inequalities (23) and (24) are in agreement with the second law of thermodynamics.

Note that, as stated in this section, inequality (23) or (24) are satisfied above the Planck time, but below the Planck time t_P , inequality (24) can be violated. In fact, for a power law cosmology with radiation-filled matter, we have $T_r \sim 1/a(t)$ and $a(t) \sim \sqrt{t}$. Hence $T_r E_{ms}^r \sim \sqrt{t}$ and (24) is violated below $t \simeq t_P$. However, we expect that the spacetime cannot more be depicted as a classical manifold but rather that the non-commutative quantum structure of the spacetime emerges (see [32, 33, 34, 35, 36, 37, 38,

39]). In this frame, it is expected that a minimum volume does appear (see in particular [37, 33, 34]) of the order of the Planck volume and as a result the inequalities is not violated. This implies that a quantum spacetime conspires below the Planck time to save the second law of thermodynamics for a non-phantom filled universe. This implies that for spatimes filled with a non phantom matter content, the second law is always satisfied.

7 Discussion and final remarks

In this paper we have investigated some interesting thermodynamical consequences of our new recently proposed formula for the Bekenstein-Hawking entropy in expanding universes. This formula, according to the holographic principle, has been applied to the apparent horizon of our visible universe. We showed that only flat Friedmann universes have zero internal energy. The introduction of a non-vanishing spatial curvature k inhibits the conservation of the total energy of the universe. This can be explained by the fact that, thanks to the Friedmann equations, to a non-zero spatial curvature can be associated a binding energy. For an expanding universe with $k = -1$ new space is created with positive energy, while for $k = +1$ with negative energy (attractive gravity).

Another interesting and useful result of this paper is the one of section 5 regarding the free Helmholtz energy F_h : the free energy of the universe is stationary only when a de Sitter phase is reached. The first implication is that to a cosmological constant of an expanding universe for comoving observers can be associated a non-vanishing temperature that in turn is the mean temperature of the universe during a pure de Sitter phase. Since the free energy is stationary only when a de Sitter phase is reached, we may think our universe as evolving between two equilibrium phases. Hence, the actual small cosmological constant allows the universe to evolve up to an asymptotically stable equilibrium configuration. Also note that in this new view, the primordial inflation happens because the temperature T_u of the universe dropped more quickly with respect to T_h , and as a consequence a pure de Sitter phase begins when $T_u = T_h$. This interesting point of view has been partially addressed at the appendix of [21]. Our view can have some similarity with the construction made in [40], where the Einstein's equations have been written in such a way that the non-static nature of a given spacetime with respect to a suitable local reference frame is due to the difference between the degrees of freedom N_{sur} of the boundary of a given bulk region and the ones of a bulk N_{bulk} . In this context, the degrees of

freedom are computed in terms of the average Davies-Unruh temperature T_{avg} at the boundary of a bulk region. Although some interesting analogy can exist (instead of $N_{sur} - N_{bulk}$ we use $T_u - T_h$ to describe the thermal history of our universe), we refer to the temperature of the bulk T_u as the temperature of the matter-energy content inside L_h at the present time, (practically the temperature of the CMBR), while N_{bulk} in [40] is defined in terms of T_{avg} . In our view the difference $T_u - T_h$ is relevant for the dynamics of the proper areal radius L_h . A de Sitter universe with $T_u = T_h$ has an apparent horizon with stationary proper area, but the universe is still expanding for a non local reference frame (comoving observers). Moreover, in [40] it is assumed the usual equipartition law, but this is strictly valid for systems whose Hamiltonian is quadratic in the conjugate variables and in the classical range where the fluid is described in the canonical ensemble by the Boltzmann distribution.

Further interesting physical consequences are due to the inequalities (23) and (24). The inequality (23) in particular can be put in a form similar to a quantum uncertainty relation for the temperature T_u and the proper radius L_h of our universe as

$$T_u L_h \geq \frac{\hbar c}{4\pi k_B}. \quad (25)$$

Since at first look the quantities at the left member of (25) have nothing to do with the fundamental constant \hbar , the presence of \hbar in (25) is rather intriguing. For our universe, homogeneous on scales $> 70 - 100$ Mpc, we can consider T_u as an average quantity $\langle T_u \rangle$ calculated with respect to different inhomogeneous patches.

An important consequence of (25) (or (24)) is that a given Friedmann universe filled with non-phantom mass-energy at any fixed L_h cannot have an arbitrary small temperature. Only for expanding spacetimes without a cosmological constant we have that when $t \rightarrow \infty$, $L_h \rightarrow \infty$ and then the minimum for T_u can be zero, but only asymptotically. It should also be noticed that this result has not been obtained by the use only of Friedmann equations, but with the help of the behaviour and meaning of the free energy, equation (19). A standard result, obtained after writing the Friedmann equations as a first law with $S_h \sim A_h/4$ but without the extra input of the free energy can be found in [41]. In this paper it is shown that, by supposing that T_h is proportional to T_u and the generalized second law holds, $T_h - T_u \leq Q$, where $Q \sim (\rho c^2 + p)$. For $c^2 \rho \geq -p$ (non-phantom matter), we have

$$T_u - T_h \geq -Q, \quad Q \geq 0. \quad (26)$$

From (26) we argue that for a non-phantom universe no positive minimum allowed value for T_u emerges. In our calculations, the inequality (25) is only a consequence of the physical meaning of the free energy. Hence, thermodynamics applied at the whole visible universe can hint us the nature of the expansion of the visible universe.

In our view the (proper area of) apparent horizon is time-increasing if and only if $T_u > T_h$, decreasing for $T_u < T_h$ and stationary for $T_u = T_h$.

A dark energy can thus be interpreted as a perfect fluid satisfying the thermodynamic relation $T_h L_h = \text{const}$ or $T_h^3 V_h = \text{const}$. According to the stationary isoentropic nature of the apparent horizon of a de Sitter expanding universe, the equality $T_h^3 V_h = \text{const}$ is nothing else but the equation of an adiabatic isoentropic transformation with $S_h \sim V_h T_h^3$.

It is important to note that inequality (25) is independent on the gravitational constant G . This fact is in line with the holographic principle where the entropy bound does not depend on G and thus can be considered as universal. This suggests the possibility that also the inequality (25) is universal and can be applied to any physical probe. We can thus formulate the following conjecture:

Any given matter-energy fluid of volume V and satisfying a non-phantom equation of state, must satisfy inequality (25).

As mentioned above, the inequality (25) can be written in terms of the volume:

$$T_u^3 V \geq \frac{1}{48\pi^2} \left(\frac{c\hbar}{k_B} \right)^3. \quad (27)$$

When non commutative effects are taken into account (see [37, 33, 34]), a minimum volume does appear and as a result a maximum temperature T_{umax} emerges of the order of the Planck temperature T_P , i.e. $T_{umax} \sim T_P/(4\pi)$ (big bang is ruled out).

Conversely, for a phantom gas the minimum attainable value (but not the maximum) for T disappears and thus the (25) or (27) can be violated. In particular, for a phantom matter probe with temperature T_{ph} and length L , we must have

$$T_{ph} L < \frac{\hbar c}{4\pi k_B}, \quad (28)$$

or the reverse of (27). Note that our proposal for a non vanishing value T_h for the temperature of the dark energy (see for example [42] and references therein) allows a non negative value for the temperature of a phantom fluid given by (28) thus solving the shortcomings of a negative temperature. Nevertheless, since $L \geq 0$, we can in principle have a phantom probe with an absolute temperature $T_{ph} < 0$ (i.e. $T_{ph} < -273.15$ C) (see [43]).

Summarizing, our conjecture implies that for a given gas in a box of typical size L there exists a critical temperature $T_c = \frac{\hbar c}{4\pi k_B L}$. Above this temperature, the gas has not a phantom-like behaviour, below T_c the system is in practice a phantom-gas and exactly at $T = T_c$ the system looks like a cosmological constant. This critical temperature depends on the size of the probe: a given gas at the temperature T can be in a phantom or non phantom phase depending on its size according to inequality (25) and (28). For a numerical example, for $L \sim 1$ meter, i.e. a box of length 1 meter, we have $T_c \simeq 0.00025K$, for $L \sim 1$ centimeter, $T_c \simeq 0.025K$, while for the whole universe (apparent horizon) $T_c \simeq 10^{-29}K$.

Obviously, independently on the correctness of the conjecture above for non cosmological systems, all these reasonings can be of interest for a better understanding on the nature of the dark energy.

As a final consideration, note that in the static case, the black hole temperature satisfies $T_{bh}M_{bh} = \hbar c^3/(8\pi Gk_B)$, i.e. in the case of a thermal equilibrium the inequality (27) becomes an equality identical to the one of a static black hole. This means that a de Sitter expanding universe at L_h has the same temperature-mass relation of a static black hole with the Misner-Sharp mass instead of the ADM mass. This is not surprising since in an expanding universe, thanks to [10, 11], a black hole can be identified by its apparent horizon rather than the teleological event horizon. In the static case the time coordinate t (as measured by the observer at spatial infinity) becomes spacelike and the spatial coordinate r timelike inside the event horizon as happens for the de Sitter expanding universe. This hints the possibility that also the interior of a static black hole can be filled with a cosmological-like dark energy at the temperature T_{bh} of its horizon. This hypothesis can be matter of future investigations.

Appendix a

McVittie [44] discovered over 80 years ago a class of solutions in the attempt to describe the metric of a spherical object embedded in an expanding Friedmann universe. The interpretation of this solution as describing a black hole in an expanding universe is currently debated (see for example [45] and references therein). Only recently [45, 46] it has been shown that the spacetime contains a surface (a null apparent horizon) that in the far future limit $t \rightarrow \infty$ and in presence of a constant positive limit for the Hubble flow H looks like a regular event horizon. The line element for the McVittie solution

can be written in isotropic coordinates as follows [46]:

$$ds^2 = -\frac{\left(1 - \frac{Gm}{2c^2 a(t)r}\right)^2}{\left(1 + \frac{Gm}{2c^2 a(t)r}\right)^2} c^2 dt^2 + a(t)^2 \left(1 + \frac{Gm}{2c^2 a(t)r}\right)^4 (dr^2 + r^2 d\Omega^2), \quad (29)$$

where m is a mass parameter and $H(t) = a_{,t}(t)/a(t)$. For the solution (29) we have $K_a^a = 3H$ that is obviously constant on comoving foliations and the conditions to apply formula (3) are fulfilled. Moreover, for the density $\rho(t)$ we have the usual Friedmann equation $H^2 = 8\pi G\rho/3$. We can investigate the relation between the mass parameter m of (29) with the mass excess parameter δM present in (3). To start with, note that the McVittie solution reduces to a Friedmann flat universe for $m = 0$ and as a consequence we limit to the case $k = 0$ in (3). By inverting the inequality (3), we have a sufficient condition for the formation of trapped surfaces. After denoting with M_d

$$\frac{G}{c^2} M_d = \frac{L}{2} + \frac{AH}{4\pi c}, \quad (30)$$

this condition becomes $\delta M \geq M_d$. Remember that δM denotes the proper mass excess necessary to form a trapped surface. At this stage of the tractation, the Bekenstein bound comes in action, i.e. $S \leq S_{max} = \frac{2\pi k_B L E}{\hbar c}$. In the usual tractation, E is the energy related to the mass concentration leading to a black hole. In the static context, the term with H is vanishing in (30) and inequality $\delta M \geq M_d$ is saturated with the ADM mass m_{adm} with $m_{adm} = c^2 L / (2G)$. In a cosmological context, where we have not an asymptotically flat reference metric, the situation is rather more complicated and it is not clear what kind of energy E is suitable for the Bekenstein bound. The theorems quoted in [22, 23, 24, 25] suggest that in a cosmological context the correct energy to put in the Bekenstein bound is the ('dynamical one') $E = M_d c^2$.

The McVittie solution (29) has apparent horizons at the proper areal radius $L = ra(t) \left(1 + \frac{Gm}{2c^2 a(t)r}\right)^2$ given by the roots of the equation involving ingoing geodesics:

$$\frac{H^2 L^3}{c^2} - L + \frac{2Gm}{c^2} = 0. \quad (31)$$

Equation (3) has roots depending on the discriminant $D = -1/(27H^6) + m^2/(4H^4)$. Only for $D \leq 0$ we have positive roots. We denote with L_A the smaller positive root. The surface $L = L_A$ at a fixed time is a spacelike apparent horizon that for $t \rightarrow \infty$ and for $H \rightarrow H_0 = const > 0$ in this limit becomes the future event horizon and a black hole emerges.

The mass parameter m in (29) is not the active gravitational mass. The active gravitational mass M_A calculated at L_A is given by $M_A = m + \rho V_A$, where $V_A = 4\pi L_A^3/3$ and hence by the equation $H^2 = 8\pi G\rho/3$ we have

$$M_A = m + \frac{H^2 L_A^3}{2G}, \quad (32)$$

and thanks to the (31) we get $M_A = \frac{c^2 L_A}{2G}$, i.e. M_A is nothing else but the Misner-Sharp mass $M_{ms} = M_A$. As a consequence for the McVittie solution (29) we have

$$M_d = M_{ms} + \frac{cH}{G} L_A^2. \quad (33)$$

According to (3), to form a black hole in an expanding universe it requires a proper mass concentration M_d greater than the Misner-Sharp mass M_{ms} ² by a term depicting the dynamical nature of a Friedmann cosmological background. In practice, for spherical objects with areal radius L_A embedded in a Friedmann flat universe, the Bekenstein entropy bound can be generalized to encompass the dynamical degrees of freedom due to the expanding universe in the following way:

$$S \leq S_{max} = \frac{2\pi k_B L_A}{\hbar c} \left[E_{ms} + \frac{c^3 H}{G} L_A^2 \right], \quad (34)$$

with L_A the areal radius of the spherical region and $E_{ms} = M_{ms}c^2$ the Misner-Sharp energy of a black hole at its apparent horizon L_A .

As a final consideration, we could apply the technology of paper [9] to calculate the first law of thermodynamics for (29) at L_A . We obtain in particular an expression for the internal energy U_A of (29) at $L = L_A$ given by

$$U_{A,t} = \frac{c^4}{2G} L_{A,t} + H_{,t} \left[\frac{c^3}{2G} L_A^2 + \frac{4\pi}{c} L_A^4 T_A^{(2)} \right], \quad (35)$$

where $T_A^{(2)}$ is the trace of the energy-stress tensor on the normal space $\{0, 1\}$ of (29) calculated at L_A . It can be interesting to note that in presence of a positive cosmological constant where the spacelike apparent horizon L_A becomes for $t \rightarrow \infty$ the future event horizon, the internal energy is conserved in such a limit, i.e. the internal energy of the teleological McVittie black hole is constant in time. Since for $m = 0$ ³ the (29) reduces to

²Note that the Misner-Sharp mass is expected to include the negative gravitational energy inside L_A .

³As shown in [9], the internal energy of a Friedmann flat universe equipped with a positive cosmological constant at the apparent horizon is zero.

the Friedmann flat metric and that in presence of a positive cosmological constant Λ $L_A(t \rightarrow \infty) \sim 1/\sqrt{\Lambda}$, we conclude that the total internal energy U_A for the McVittie black hole is $U_A = \gamma m c^2$, $\gamma \in R^+$ and not $M_{ms} c^2$. By choosing $\gamma = 1$, we can identify the mass parameter m as the total internal energy of the teleological McVittie black hole.

Appendix b

In this appendix we study the partition function in the flat case, where a simple expression for F_h is available. In this case we have

$$F_h = -T_h S_h = -\frac{3c^4}{4G} L_h = -\frac{3c^5}{4GH}. \quad (36)$$

To start with, we must formally introduce the partition function Z . Suppose that the spacetime is equipped with a set $\{s\}$ of countable microstates.

The partition function can thus be obtained in the standard way by means of the expression $S = -k_B \sum_s p_s \ln p_s$ together with the constraints $\sum_s p_s = 1$, $\langle U \rangle = \sum_s p_s U_s$, where p_s is the probability and U_s the energy of the microstate 's'. Thanks to usual relations and the technique of Langrangian multipliers we get

$$Z = e^{-\frac{F}{k_B T}} = \sum_s e^{-\frac{U_s}{k_B T}}, \quad p_s = \frac{e^{-\frac{U_s}{k_B T}}}{Z}, \quad \langle U \rangle = \frac{\sum_s e^{-\frac{U_s}{k_B T}} U_s}{Z}. \quad (37)$$

In the flat case we found $\langle U_h \rangle = 0$. From (37) we must have

$$\langle U_h \rangle = \frac{\sum_s e^{-\frac{U_{hs}}{k_B T_h}} U_{hs}}{Z} = 0. \quad (38)$$

The sum in (38) can be made zero by supposing that for any microstate 'i' with energy $U_{hi} > 0$, there exists a microstate 'j' with energy $U_{hj} < 0$ such that $z_i U_{hi} = -z_j U_{hj}$ where $z_k = e^{-U_{hk}/k_B T_h}$, but we have no arguments supporting this condition. Hence, the most simple and conservative hypothesis is to set $\forall i \in \{s\}, U_{hi} = 0$. As a consequence, thanks to (36) and (37) we have

$$S_h = k_B \ln Z, \quad Z = \sum_s \text{microstates}. \quad (39)$$

We can measure the proper radius $L_h = c/H$ of the apparent horizon in terms of the fundamental Planck length L_P , i.e. $L_h/L_P = N_P$. We obtain

$S_h = k_B 3\pi N_P^2$ and

$$Z = \sum_s \text{microstates} = e^{3\pi N_P^2}. \quad (40)$$

According to the formulas (39) and (40), the Gibbs factor is exactly 1 and as a consequence the expression for S_h reduces to the one of the microcanonical ensemble.

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